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# Generalized Nonlinear Wave Equation in Frequency Domain

Hairun Guo\*, Xianglong Zeng and Morten Bache

*Group of ultrafast nonlinear optics, Department of photonics engineering (DTU Fotonik), Technical University of Denmark, DK-2800, Kgs. Lyngby, Denmark*

*\* Corresponding author: haig@fotonik.dtu.dk*

We interpret the forward Maxwell equation with up to third order induced polarizations and get so called nonlinear wave equation in frequency domain (NWEF), which is based on Maxwell wave equation and using slowly varying spectral amplitude approximation. The NWEF is generalized in concept as it directly describes the electric field dynamics rather than the envelope dynamics and because it concludes most current-interested nonlinear processes such as three-wave mixing, four-wave-mixing and material Raman effects. We give two sets of NWEF, one is a 1+1D equation describing the (approximated) planar wave propagation in nonlinear bulk material and the other corresponds to the propagation in a waveguide structure. © 2013 Optical Society of America

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## Introduction

There are three parts in the derivation. First, we review the derivation of Maxwell's wave equation in frequency domain and the expression of nonlinear induced polarization. Second, in the approximation of planar wave propagation, 1+1D nonlinear wave equation in frequency domain (NWEF) is derived as an interpretation of the 1+1D forward Maxwell equation with up to third order induced polarizations. Third, considering a waveguide structure, NWEF dependent on spatial mode profile is derived.

### 1. Maxwell's wave equation and nonlinear induced polarization

We start from Maxwell's equations and material equations shown below [1–3]:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \nabla \times \mathbf{H} = -\frac{\partial \mathbf{D}}{\partial t} + \mathbf{J} \quad (1)$$

$$\nabla \bullet \mathbf{B} = 0, \nabla \bullet \mathbf{D} = \rho, \nabla \bullet \mathbf{J} = -\frac{\partial \rho}{\partial t} \quad (2)$$

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}, \mathbf{B} = \mu_0 \mathbf{H} + \mathbf{M}, \mathbf{J} = \sigma \mathbf{E} \quad (3)$$

" $\nabla$ " is the Laplace operator,  $\mathbf{E}$  and  $\mathbf{H}$  indicate the electric field (unit:  $\frac{V}{m}$ ) and magnetic field (unit:  $\frac{A}{m}$ ) vectors,  $\mathbf{D}$  and  $\mathbf{B}$  indicate the electric and magnetic flux densities.  $\mathbf{J}$  is the current density vector and  $\rho$  is the charge density, representing the sources for the electromagnetic field.  $\varepsilon_0$  is the vacuum permittivity (unit:  $\frac{F}{m}$ ) and  $\mu_0$  is the vacuum permeability (unit:  $\frac{H}{m}$ ).  $\mathbf{P}$  and  $\mathbf{M}$  indicate the induced electric and magnetic polarization.

In the absence of free charges in a nonmagnetic medium, we have  $\mathbf{M} = 0$ ,  $\mathbf{J} = 0$  and  $\rho = 0$ . Hence the Maxwell's equations is simplified to:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} \quad (4)$$

After Fourier transforming the above equations ( $\tilde{F}(\omega) = \int_{-\infty}^{\infty} F(t)e^{-i\omega t} dt$ , matching the DFT algorithm in @MATLAB), we have:

$$\nabla \times \tilde{\mathbf{E}} = -i\omega \tilde{\mathbf{B}} = -i\omega \mu_0 \tilde{\mathbf{H}}, \nabla \times \tilde{\mathbf{H}} = i\omega \tilde{\mathbf{D}} = i\omega(\varepsilon_0 \tilde{\mathbf{E}} + \tilde{\mathbf{P}}) \quad (5)$$

To derive the wave equation regarding the electric field, we use the relation:

$$\nabla \times (\nabla \times \tilde{\mathbf{E}}) = \nabla(\nabla \bullet \tilde{\mathbf{E}}) - \nabla^2 \tilde{\mathbf{E}} \quad (6)$$

in which the left side can be extended as:

$$\begin{aligned} \nabla \times (\nabla \times \tilde{\mathbf{E}}) &= \nabla \times (-i\omega \mu_0 \tilde{\mathbf{H}}) = -i\omega \mu_0 \cdot (\nabla \times \tilde{\mathbf{H}}) \\ &= -i\omega \mu_0 \cdot i\omega(\varepsilon_0 \tilde{\mathbf{E}} + \tilde{\mathbf{P}}) = \omega^2 \mu_0 \varepsilon_0 \tilde{\mathbf{E}} + \omega^2 \mu_0 \tilde{\mathbf{P}} = k_0^2(\omega) \tilde{\mathbf{E}} + \omega^2 \mu_0 \tilde{\mathbf{P}} \end{aligned} \quad (7)$$

On the right side, we have:

$$\nabla(\nabla \bullet \tilde{\mathbf{E}}) - \nabla^2 \tilde{\mathbf{E}} \approx -\nabla^2 \tilde{\mathbf{E}} \quad (8)$$

which is supported if: 1) the high order induced electric polarizations (nonlinear induced polarizations) are considered perturbations to the first order induced polarization (linear induced polarization), i.e.  $\tilde{\mathbf{P}} = \tilde{\mathbf{P}}_L + \tilde{\mathbf{P}}_{NL} \approx \tilde{\mathbf{P}}_L = 2\pi\varepsilon_0\tilde{\chi}^{(1)}\tilde{\mathbf{E}}$ ; 2) the relative permittivity  $\varepsilon_r = 1 + 2\pi\tilde{\chi}^{(1)}$  is independent on the spatial distribution, i.e.  $\nabla \bullet \tilde{\mathbf{E}} = \nabla \bullet \tilde{\mathbf{D}}/\varepsilon_0\varepsilon_r = 0$ .

Therefore, in frequency domain, we have the Maxwell's wave equation for the electric field:

$$\nabla^2 \tilde{\mathbf{E}} + k_0^2(\omega) \tilde{\mathbf{E}} + \omega^2 \mu_0 \tilde{\mathbf{P}} = 0 \quad (9)$$

In time-domain, it is:

$$\nabla^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} - \mu_0 \frac{\partial^2 \mathbf{P}}{\partial t^2} = 0 \quad (10)$$

The induced polarization is further expressed as:

$$\mathbf{P} = \mathbf{P}^{(1)} + \mathbf{P}^{(2)} + \mathbf{P}^{(3)} + \dots + \mathbf{P}^{(m)} \quad (11)$$

where the generalized expression of both the linear induced polarization  $\mathbf{P}^{(1)} = \mathbf{P}_L$  and the nonlinear induced polarization  $\mathbf{P}^{(m)}$  is:

$$\begin{aligned} \mathbf{P}^{(m)} &= \varepsilon_0 \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \dots \int_{-\infty}^{\infty} dt_m \chi^{(m)}(t_1, t_2, \dots, t_m) |\mathbf{E}(t-t_1) \mathbf{E}(t-t_2) \dots \mathbf{E}(t-t_m)| \\ &= \varepsilon_0 \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \dots \int_{-\infty}^{\infty} dt_m \chi^{(m)}(t-t_1, t-t_2, \dots, t-t_m) |\mathbf{E}(t_1) \mathbf{E}(t_2) \dots \mathbf{E}(t_m)| \\ &= \varepsilon_0 \int_{-\infty}^{\infty} d\omega_1 \int_{-\infty}^{\infty} d\omega_2 \dots \int_{-\infty}^{\infty} d\omega_m \tilde{\chi}^{(m)}(\omega_1, \omega_2, \dots, \omega_m) |\tilde{\mathbf{E}}(\omega_1) \tilde{\mathbf{E}}(\omega_2) \dots \tilde{\mathbf{E}}(\omega_m)| e^{it \sum \omega_i} \end{aligned} \quad (12)$$

where:

$$\tilde{\chi}^{(m)}(\omega_1, \omega_2, \dots, \omega_m) = \frac{1}{(2\pi)^m} \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \dots \int_{-\infty}^{\infty} dt_m \chi^{(m)}(t_1, t_2, \dots, t_m) e^{-i \sum \omega_i t_i} \quad (13)$$

$\chi^{(m)}$  is the temporal response function of the material, also called susceptibility in frequency domain, which is a (m+1)-rank tensor. The calculations among the electric fields are dyadic product which result in an m-rank tensor. "||" indicate the multiple tensor product between two tensors, i.e. a (m+1)-rank tensor

and a m-rank tensor (or dyadic tensor). Therefore, the induced polarization  $\mathbf{P}^{(m)}$  is a vector. In frequency domain, it has:

$$\begin{aligned}\tilde{\mathbf{P}}^{(m)}(\omega) &= \int_{-\infty}^{+\infty} \mathbf{P}^{(m)} e^{-i\omega t} dt \\ &= \varepsilon_0 \int_{-\infty}^{\infty} d\omega_1 \cdots \int_{-\infty}^{\infty} d\omega_m \tilde{\chi}^{(m)}(\omega_1, \dots, \omega_m) |\tilde{\mathbf{E}}(\omega_1) \cdots \tilde{\mathbf{E}}(\omega_m)| \int_{-\infty}^{+\infty} e^{-i(\omega - \sum \omega_i)t} dt \\ &= 2\pi\varepsilon_0 \int_{-\infty}^{\infty} d\omega_1 \cdots \int_{-\infty}^{\infty} d\omega_m \tilde{\chi}^{(m)}(\omega_1, \dots, \omega_m) |\tilde{\mathbf{E}}(\omega_1) \cdots \tilde{\mathbf{E}}(\omega_m)| \delta(\omega - \sum \omega_i) \quad (14)\end{aligned}$$

Here, the delta function implies that the induced polarization always corresponds to the frequency which equals to the sum of the frequencies of the contributing electric fields.

If setting  $\Omega = \sum \omega_i$ , i.e.  $d\Omega = d\omega_i$  and  $\int_{-\infty}^{+\infty} \delta(\omega - \Omega) d\Omega = 1$ , the above equation becomes:

$$\begin{aligned}\tilde{\mathbf{P}}^{(m)}(\Omega) &= 2\pi\varepsilon_0 \int_{-\infty}^{\infty} d\omega_1 \int_{-\infty}^{\infty} d\omega_2 \cdots \int_{-\infty}^{\infty} d\omega_{m-1} \\ &\quad \times \tilde{\chi}^{(m)}(\omega_1, \omega_2, \dots, \Omega - \sum_{i=1}^{m-1} \omega_i) |\mathbf{E}(\omega_1) \mathbf{E}(\omega_2) \cdots \mathbf{E}(\Omega - \sum_{i=1}^{m-1} \omega_i)|, m \geq 2 \quad (15)\end{aligned}$$

and  $\tilde{\mathbf{P}}^{(1)}(\Omega) = 2\pi\varepsilon_0 \tilde{\chi}^{(1)}(\Omega) \bullet \tilde{\mathbf{E}}(\Omega)$ .

Since  $\mathbf{P}^{(m)}$  is a vector, it can be written as a sum of its components, each casting to one dimension, i.e.  $\mathbf{P}^{(m)} = \sum \hat{j} P_j^{(m)}$ . Analogously, The nonlinear response tensor  $\chi^{(m)}$  has  $\chi^{(m)} = \sum \hat{j} \mathbf{R}_j^{(m)}$ . Now  $\mathbf{R}_j^{(m)}$  is an m-rank tensor and has  $\mathbf{R}_j^{(m)} = \sum_{\alpha_1 \cdots \alpha_m} \left[ \left( \prod_{s=1}^m \hat{\alpha}_s \right) \cdot \chi_{j;\alpha_1 \cdots \alpha_m}^{(m)} \right]$ , where  $j, \alpha_1, \dots, \alpha_m$  is dimension mark.

Then, the component of the induced polarization  $P_j^{(m)}$  is:

$$\begin{aligned}P_j^{(m)}(t) &= \varepsilon_0 \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \cdots \int_{-\infty}^{\infty} dt_m \mathbf{R}_j^{(m)}(t - t_1, t - t_2, \dots, t - t_m) |\mathbf{E}(t_1) \mathbf{E}(t_2) \cdots \mathbf{E}(t_m)| \\ &= \varepsilon_0 \sum_{\alpha_1 \cdots \alpha_m} \left\{ \int_{-\infty}^{\infty} dt_1 \cdots \int_{-\infty}^{\infty} dt_m \chi_{j;\alpha_1 \cdots \alpha_m}^{(m)}(t - t_1, \dots, t - t_m) \cdot E_{\alpha_1}(t_1) \cdots E_{\alpha_m}(t_m) \right\} \quad (16)\end{aligned}$$

In frequency domain, it is:

$$\begin{aligned}\tilde{P}_j^{(m)}(\Omega) &= 2\pi\varepsilon_0 \sum_{\alpha_1 \cdots \alpha_m} \left\{ \int_{-\infty}^{\infty} d\omega_1 \cdots \int_{-\infty}^{\infty} d\omega_{m-1} \tilde{\chi}_{j;\alpha_1 \cdots \alpha_m}^{(m)}(\omega_1, \dots, \omega_{m-1}, \Omega - \sum_{i=1}^{m-1} \omega_i) \right. \\ &\quad \left. \times \tilde{E}_{\alpha_1}(\omega_1) \cdots \tilde{E}_{\alpha_{m-1}}(\omega_{m-1}) \tilde{E}_{\alpha_m}(\Omega - \sum_{i=1}^{m-1} \omega_i) \right\}, m \geq 2 \quad (17)\end{aligned}$$

and  $\tilde{P}_j^{(1)}(\Omega) = 2\pi\varepsilon_0 \sum_{\alpha_1} \left\{ \tilde{\chi}_{j;\alpha_1}^{(1)}(\Omega) \tilde{E}_{\alpha_1}(\Omega) \right\}$ .  $\tilde{\chi}_{j;\alpha_1 \cdots \alpha_m}^{(m)}$  corresponds to one component of the tensor  $\tilde{\chi}^{(m)}$ , in which  $\tilde{\chi}_{j;\alpha_1}^{(1)}$  is coming from the matrix  $\tilde{\chi}^{(1)}$ .

Therefore, each component of the electric field  $\tilde{E}_j$  has a wave equation which, in frequency domain, is written as:

$$\nabla^2 \tilde{E}_j + k_0^2(\omega) \tilde{E}_j + \omega^2 \mu_0 \tilde{P}_j = 0; \tilde{P}_j = \tilde{P}_j^{(1)} + \tilde{P}_j^{(2)} + \cdots + \tilde{P}_j^{(m)} \quad (18)$$

In particular, in uniaxial and biaxial crystals as well as cubic/isotropic materials, matrix  $\tilde{\chi}^{(1)}$  only has diagonal elements and therefore  $\tilde{P}_j^{(1)}(\Omega) = 2\pi\varepsilon_0 \tilde{\chi}_{j;j}^{(1)}(\Omega) \tilde{E}_j(\Omega)$ . we combine  $\tilde{P}_j^{(1)}$  with  $k_0^2 \tilde{E}_j$  and update the wave equation as:

$$\nabla^2 \tilde{E}_j + k_j^2(\omega) \tilde{E}_j + \omega^2 \mu_0 \tilde{P}_{j,NL} = 0; \tilde{P}_{j,NL} = \tilde{P}_j^{(2)} + \cdots + \tilde{P}_j^{(m)} \quad (19)$$

where  $k_j^2(\omega) = k_0^2(1 + 2\pi\tilde{\chi}_{j;j}^{(1)})$  and the refractive index in the dimension  $j$  is therefore defined as  $n_j = \sqrt{1 + 2\pi\tilde{\chi}_{j;j}^{(1)}}$ .

## 2. 1+1D NWEF

In the approximation of planer wave propagation, we neglect the spacial dynamics in the propagation of the electric field but focus on the temporal and spectral dynamics. the Laplace operator is therefore reduced to the only derivative with respect to propagation axis  $z$  since the spacial dynamics is eliminated, i.e.  $\nabla^2 \rightarrow \frac{\partial^2}{\partial z^2}$ . Therefore, Eq.(19) is reduced to a 1+1D wave equation:

$$\frac{\partial^2}{\partial z^2} \tilde{E}_j + k_j^2(\omega) \tilde{E}_j + \omega^2 \mu_0 \tilde{P}_{j,NL} = 0 \quad (20)$$

By factoring out the fast dependence of the propagation coordinate from the electric field for all the frequencies, i.e.  $\tilde{E}_j(z, \omega) = \tilde{A}_j(z, \omega) e^{-ik_j(\omega)z}$ , we get:

$$\frac{\partial^2}{\partial z^2} \tilde{A}_j - 2ik_j(\omega) \frac{\partial}{\partial z} \tilde{A}_j + \omega^2 \mu_0 \tilde{P}_{j,NL} e^{ik_j z} = 0 \quad (21)$$

In the slowly varying spectral amplitude approximation (SVSAA) [4], i.e.  $\left| \frac{\partial}{\partial z} \tilde{A}_j \right| \ll \left| k_j \tilde{A}_j \right|$ , we have  $\frac{\partial^2}{\partial z^2} \tilde{A}_j \ll \frac{\partial}{\partial z} k_j \tilde{A}_j$ , which means the second-order derivative can be removed.

So, we have:

$$\frac{\partial}{\partial z} \tilde{A}_j = -i \frac{\omega^2 \mu_0}{2k_j(\omega)} \tilde{P}_{j,NL} e^{ik_j z} \quad (22)$$

and the reduced 1+1D Maxwell equation (also called forward Maxwell equation) regarding the electric field [5–7]:

$$\frac{\partial \tilde{E}_j}{\partial z} + ik_j(\omega) \tilde{E}_j = -i \frac{\omega^2 \mu_0}{2k_j(\omega)} \tilde{P}_{j,NL} \quad (23)$$

Among all the nonlinear induced polarizations, second-order and third order nonlinear induced polarizations are most concerned. For second-order nonlinear induced polarization, the response is always assumed instantaneous, giving rise to three-wave-mixing (TWM). The response function has  $\chi_{j;\alpha_1\alpha_2}^{(2)}(t_1, t_2) = \bar{\chi}_{j;\alpha_1\alpha_2}^{(2)} \delta(t_1) \delta(t_2)$ , where  $\bar{\chi}_{j;\alpha_1\alpha_2}^{(2)}$  is a constant indicating the response intensity. Correspondingly, in frequency domain, the susceptibility has  $\tilde{\chi}_{j;\alpha_1\alpha_2}^{(2)}(\omega_1, \omega_2) = \frac{1}{(2\pi)^2} \bar{\chi}_{j;\alpha_1\alpha_2}^{(2)}$  according to Eq.(13), i.e. constant for all the frequencies  $\omega_1$  and  $\omega_2$ . Hence, the second-order nonlinear induced polarization  $P_j^{(2)}$  has the same form as shown in [8, 9]:

$$P_j^{(2)}(t) = \varepsilon_0 \sum_{\alpha_1\alpha_2} \left\{ \bar{\chi}_{j;\alpha_1\alpha_2}^{(2)} E_{\alpha_1} E_{\alpha_2} \right\} \quad (24)$$

$$\tilde{P}_j^{(2)}(\omega) = \varepsilon_0 \sum_{\alpha_1\alpha_2} \left\{ \bar{\chi}_{j;\alpha_1\alpha_2}^{(2)} \tilde{E}_{\alpha_1} \frac{\otimes}{2\pi} \tilde{E}_{\alpha_2} \right\} = \varepsilon_0 \sum_{\alpha_1\alpha_2} \left\{ \bar{\chi}_{j;\alpha_1\alpha_2}^{(2)} F[E_{\alpha_1} E_{\alpha_2}] \right\} \quad (25)$$

As for third-order nonlinear induced polarization, the response is not fully instantaneous but consists of the instantaneous electronic Kerr response and a fraction of non-instantaneous vibrational Raman response. The response function is therefore written as  $\chi_{j;\alpha_1\alpha_2\alpha_3}^{(3)}(t_1, t_2, t_3) = \bar{\chi}_{j;\alpha_1\alpha_2\alpha_3}^{(3)} R(t_1) \delta(t_2 - t_1) \delta(t_3)$ , where  $R(t) = (1 - f_R) \delta(t) + f_R h_R(t)$  and  $\int_{-\infty}^{\infty} h_R(t) dt = 1$ .  $f_R$  indicates the amount of Raman fraction.  $h_R(t)$  is the temporal Raman response function. The remaining instantaneous response is called electronic Kerr response which is the origin of the effects of self-phase modulation (SPM), cross-phase modulation (XPM) and four-wave-mixing (FWM). In frequency domain, the susceptibility is  $\tilde{\chi}_{j;\alpha_1\alpha_2\alpha_3}^{(3)}(\omega_1, \omega_2, \omega_3) = \frac{1}{(2\pi)^3} \bar{\chi}_{j;\alpha_1\alpha_2\alpha_3}^{(3)} \left[ (1 - f_R) + f_R \tilde{h}_R(\omega_1 + \omega_2) \right]$ . Hence, the third-order nonlinear induced polarization  $P_j^{(3)}$  has:

$$P_j^{(3)}(t) = \varepsilon_0 \sum_{\alpha_1 \alpha_2 \alpha_3} \left\{ \bar{\chi}_{j;\alpha_1 \alpha_2 \alpha_3}^{(3)} [(1 - f_R) E_{\alpha_1} E_{\alpha_2} E_{\alpha_3} + f_R (h_R \otimes (E_{\alpha_1} E_{\alpha_2})) E_{\alpha_3}] \right\} \quad (26)$$

$$\begin{aligned} \tilde{P}_j^{(3)}(\omega) &= \varepsilon_0 \sum_{\alpha_1 \alpha_2 \alpha_3} \left\{ \bar{\chi}_{j;\alpha_1 \alpha_2 \alpha_3}^{(3)} \left[ (1 - f_R) (\tilde{E}_{\alpha_1} \otimes_{2\pi} \tilde{E}_{\alpha_2} \otimes_{2\pi} \tilde{E}_{\alpha_3}) + f_R (\tilde{h}_R (\tilde{E}_{\alpha_1} \otimes_{2\pi} \tilde{E}_{\alpha_2})) \otimes_{2\pi} \tilde{E}_{\alpha_3} \right] \right\} \\ &= \varepsilon_0 \sum_{\alpha_1 \alpha_2 \alpha_3} \left\{ \bar{\chi}_{j;\alpha_1 \alpha_2 \alpha_3}^{(3)} \left[ (1 - f_R) F [E_{\alpha_1} E_{\alpha_2} E_{\alpha_3}] + f_R F \left[ E_{\alpha_3} F^{-1} [\tilde{h}_R F [E_{\alpha_1} E_{\alpha_2}]] \right] \right] \right\} \end{aligned} \quad (27)$$

Finally, by substituting the nonlinear induced polarizations in Eq.(23) with their interpretations Eq.(25) and Eq.(27), we get the 1+1D NWEF:

$$\begin{aligned} \frac{\partial \tilde{E}_j}{\partial z} + i k_j(\omega) \tilde{E}_j &= -i \frac{\omega^2}{2c^2 k_j(\omega)} \sum_{\alpha_1 \alpha_2} \left( \bar{\chi}_{j;\alpha_1 \alpha_2}^{(2)} F [E_{\alpha_1} E_{\alpha_2}] \right) \\ &\quad - i \frac{\omega^2}{2c^2 k_j(\omega)} \sum_{\alpha_1 \alpha_2 \alpha_3} \left\{ \bar{\chi}_{j;\alpha_1 \alpha_2 \alpha_3}^{(3)} \left[ (1 - f_R) F [E_{\alpha_1} E_{\alpha_2} E_{\alpha_3}] + f_R F \left[ E_{\alpha_3} F^{-1} [\tilde{h}_R F [E_{\alpha_1} E_{\alpha_2}]] \right] \right] \right\} \end{aligned} \quad (28)$$

Equation(28) can numerically be solved by split-step Fourier method together with Runge-Kutta method. It is noted that the frequency domain has a range  $(-\infty, +\infty)$  and the contents of  $k_j(\omega)$  in negative frequencies are required. According to the causality of all the induced polarizations,  $k_j(\omega)$  shows a property of complex conjugate, i.e. its contents in negative frequencies are linked to what in positive frequencies.

### 3. NWEF in waveguide structure

In a waveguide structure, the spacial dynamics of the electric field is always dominated by the eigen modes in the waveguide, which determine the spacial distribution as well as the propagation constant of the electric field. Since the orthogonality between any of the two eigen modes, we redefine the dimension mark  $j$  in the wave equation Eq.(19) to be the mode mark. Moreover, the electric field is redefined as [10]  $\tilde{E}_j(x, y, z, \omega) = \tilde{B}_j(x, y, \omega) \tilde{A}_j^\varphi(z, \omega) = \tilde{B}_j(x, y, \omega) \tilde{A}_j(z, \omega) e^{-i\beta_j(\omega)z}$ , where  $\tilde{B}_j$  is the eigen mode distribution and  $\beta_j$  indicates the mode propagation constant. Now Eq.(19) can be expanded as:

$$\tilde{A}_j e^{-i\beta_j(\omega)z} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_j^2 - \beta_j^2 \right) \tilde{B}_j + \tilde{B}_j e^{-i\beta_j(\omega)z} \left( \frac{\partial^2}{\partial z^2} - i2\beta_j \frac{\partial}{\partial z} \right) \tilde{A}_j + \omega^2 \mu_0 \tilde{P}_{j,NL} = 0 \quad (29)$$

Remember all the eigen modes of the waveguide have  $(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_j^2 - \beta_j^2) = 0$  and using SVSAA, we get the reduced Maxwell equation for waveguide structure:

$$\frac{\partial \tilde{E}_j}{\partial z} + i\beta_j(\omega) \tilde{E}_j = -i \frac{\omega^2 \mu_0}{2\beta_j(\omega)} \tilde{P}_{j,NL} \quad (30)$$

Compared with the 1+1D forward Maxwell equation Eq.(23), the only difference in the above expression is the replacement of the spacial propagation constant  $k_j$  by mode propagation constant  $\beta_j$ . However, it should be noticed that not only the dispersion characteristics but the nonlinear induced polarizations are all revised by employing the waveguide, which will be revealed later.

Instead of focusing on the space-related electric field, now we concentrate on the spectral related amplitude  $\tilde{A}_j$  and get its dynamics by making spacial integral on both sides of Eq.(30), i.e.:

$$\iint_{\infty} dx dy \tilde{B}_j^* \left( \frac{\partial}{\partial z} + i\beta_j \right) \tilde{E}_j = -i \frac{\omega^2 \mu_0}{2\beta_j} \iint_{\infty} dx dy \tilde{B}_j^* \tilde{P}_{j,NL} \quad (31)$$

$$\frac{\partial \tilde{A}_j^\varphi}{\partial z} + i\beta_j(\omega)\tilde{A}_j^\varphi = -i\frac{\omega^2\mu_0}{2\tilde{g}_j(\omega)\beta_j(\omega)} \iint_{\infty} dxdy \tilde{B}_j^* \tilde{P}_{j,NL}, \iint_{\infty} dxdy |\tilde{B}_j|^2 = \tilde{g}_j(\omega) \quad (32)$$

Analogously, we should interpret the nonlinear induced polarizations in Eq.(32) in which the nonlinear susceptibilities are now space-dependent. For second-order nonlinear induced polarization, we employ Eq.(17) and expand it with the definition of the electric field, getting:

$$\begin{aligned} \iint_{\infty} dxdy \tilde{B}_j^* \tilde{P}_j^{(2)} &= 2\pi\epsilon_0 \sum_{\alpha_1\alpha_2-\infty}^{\infty} \int \{d\omega_1 \tilde{A}_{\alpha_1}^\varphi(\omega_1) \tilde{A}_{\alpha_2}^\varphi(\Omega - \omega_1) \\ &\times \iint_{\infty} dxdy \tilde{\chi}_{j;\alpha_1\alpha_2}^{(2)}(x, y, \omega_1, \Omega - \omega_1) \tilde{B}_j^*(x, y, \Omega) \tilde{B}_{\alpha_1}(x, y, \omega_1) \tilde{B}_{\alpha_2}(x, y, \Omega - \omega_1)\} \end{aligned} \quad (33)$$

in which we can define a spacial integral factor as Phillips et al. did in [10]:

$$\begin{aligned} \tilde{\Theta}_{j;\alpha_1\alpha_2}^{(2)}(\omega_1, \omega_2) &= \iint_{\infty} dxdy \tilde{\chi}_{j;\alpha_1\alpha_2}^{(2)}(x, y, \omega_1, \omega_2) \tilde{B}_j^*(x, y, \omega_1 + \omega_2) \tilde{B}_{\alpha_1}(x, y, \omega_1) \tilde{B}_{\alpha_2}(x, y, \omega_2) \\ &= \frac{1}{(2\pi)^2} \iint_{\infty} dxdy \tilde{\chi}_{j;\alpha_1\alpha_2}^{(2)}(x, y) \tilde{B}_j^*(x, y, \omega_1 + \omega_2) \tilde{B}_{\alpha_1}(x, y, \omega_1) \tilde{B}_{\alpha_2}(x, y, \omega_2) \end{aligned} \quad (34)$$

and simplify the expression of second-order nonlinear induced polarization as:

$$\iint_{\infty} dxdy \tilde{B}_j^* \tilde{P}_j^{(2)} = 2\pi\epsilon_0 \sum_{\alpha_1\alpha_2-\infty}^{\infty} \int d\omega_1 \tilde{A}_{\alpha_1}^\varphi(\omega_1) \tilde{A}_{\alpha_2}^\varphi(\Omega - \omega_1) \tilde{\Theta}_{j;\alpha_1\alpha_2}^{(2)}(\omega_1, \Omega - \omega_1) \quad (35)$$

Now the spacial integral factor  $\tilde{\Theta}_{j;\alpha_1\alpha_2}^{(2)}$  plays the role as the second-order susceptibility, determining the intensity of the nonlinear induced polarization. Moreover, it can be assumed that the variation of the mode distribution  $\tilde{B}_j$  with respect to frequency is slow enough compared with that of  $\tilde{A}_j^\varphi$ , or named slowly varying mode distribution approximation (SVMDA), therefore the spacial integral factor is considered constant  $\tilde{\Theta}_{j;\alpha_1\alpha_2}^{(2)}(\omega_1, \omega_2) \approx \frac{1}{(2\pi)^2} \bar{\Theta}_{j;\alpha_1\alpha_2}^{(2)}$  and the second-order induced polarization is further simplified to:

$$\iint_{\infty} dxdy \tilde{B}_j^* \tilde{P}_j^{(2)} = \epsilon_0 \sum_{\alpha_1\alpha_2} \left\{ \bar{\Theta}_{j;\alpha_1\alpha_2}^{(2)} \tilde{A}_{\alpha_1}^\varphi \otimes_{2\pi} \tilde{A}_{\alpha_2}^\varphi \right\} = \epsilon_0 \sum_{\alpha_1\alpha_2} \left\{ \bar{\Theta}_{j;\alpha_1\alpha_2}^{(2)} F[A_{\alpha_1}^\varphi A_{\alpha_2}^\varphi] \right\} \quad (36)$$

Analogously, for third-order nonlinear induced polarization, we have:

$$\iint_{\infty} dxdy \tilde{B}_j^* \tilde{P}_j^{(3)} = \epsilon_0 \sum_{\alpha_1\alpha_2\alpha_3} \left\{ \bar{\Theta}_{j;\alpha_1\alpha_2\alpha_3}^{(3)} \left[ (1 - f_R) F[A_{\alpha_1}^\varphi A_{\alpha_2}^\varphi A_{\alpha_3}^\varphi] + f_R F[A_{\alpha_3}^\varphi F^{-1}[\tilde{h}_R F[A_{\alpha_1}^\varphi A_{\alpha_2}^\varphi]]] \right] \right\} \quad (37)$$

The spacial integral factor playing a role as the third-order susceptibility is:

$$\begin{aligned} \tilde{\Theta}_{j;\alpha_1\alpha_2\alpha_3}^{(3)}(\omega_1, \omega_2, \omega_3) &= \iint_{\infty} dxdy \tilde{\chi}_{j;\alpha_1\alpha_2\alpha_3}^{(3)}(x, y, \omega_1, \omega_2, \omega_3) \tilde{B}_j^*(x, y, \sum_n \omega_n) \tilde{B}_{\alpha_1}(x, y, \omega_1) \tilde{B}_{\alpha_2}(x, y, \omega_2) \tilde{B}_{\alpha_3}(x, y, \omega_3) \\ &= \frac{[1 - f_R + f_R \tilde{h}_R(\omega_1 + \omega_2)]}{(2\pi)^3} \iint_{\infty} dxdy \tilde{\chi}_{j;\alpha_1\alpha_2\alpha_3}^{(3)} \tilde{B}_j^*(x, y, \sum_n \omega_n) \tilde{B}_{\alpha_1}(x, y, \omega_1) \tilde{B}_{\alpha_2}(x, y, \omega_2) \tilde{B}_{\alpha_3}(x, y, \omega_3) \\ &\approx \frac{1}{(2\pi)^3} \bar{\Theta}_{j;\alpha_1\alpha_2\alpha_3}^{(3)} [1 - f_R + f_R \tilde{h}_R(\omega_1 + \omega_2)] \end{aligned} \quad (38)$$

Finally, by substituting the nonlinear induced polarizations in Eq.(32) with their interpretations Eq.(36) and Eq.(37), we get the NWEF in the waveguide structure:

$$\begin{aligned} \frac{\partial \tilde{A}_j^\varphi}{\partial z} + i\beta_j(\omega)\tilde{A}_j^\varphi = & -i\frac{\omega^2}{2c^2\tilde{g}_j(\omega)\beta_j(\omega)} \sum_{\alpha_1\alpha_2} \left\{ \bar{\Theta}_{j;\alpha_1\alpha_2}^{(2)} F[A_{\alpha_1}^\varphi A_{\alpha_2}^\varphi] \right\} \\ & - i\frac{\omega^2}{2c^2\tilde{g}_j(\omega)\beta_j(\omega)} \sum_{\alpha_1\alpha_2\alpha_3} \left\{ \bar{\Theta}_{j;\alpha_1\alpha_2\alpha_3}^{(3)} \left[ (1-f_R)F[A_{\alpha_1}^\varphi A_{\alpha_2}^\varphi A_{\alpha_3}^\varphi] + f_RF[A_{\alpha_3}^\varphi F^{-1}[\tilde{h}_RF[A_{\alpha_1}^\varphi A_{\alpha_2}^\varphi]]] \right] \right\} \end{aligned} \quad (39)$$

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